This exercise sheet is **optional**. It is meant to give you an opportunity to revise the basic definitions of probability theory and provide you with some additional tasks which should prove insightful. These are not relevant to the course and are marked with a (*). They are not meant to intimidate, but rather to provide something worthwhile for your further academic career, should you be so inclined as to pursue a field of study which relies on probability theory, or in general interested in some of the details of the course. All the other tasks are essential for the course, and you should understand them and be able to reproduce them in an exam.

**Definitions**

Write down the definitions of the following items:

- Probabilities with their three defining properties normalization, additivity, and nonnegativity.
- The marginal probability of an event $x$.
- The conditional probability with prior $y$ and posterior $x$.
- Bayes’ theorem.
- Expectation and variance of a random variable $X$.
- Statistical independence of two random variables $X$ and $Y$.
- The covariance of two random variables $X$ and $Y$.

**Some Implications**

Prove:

- Bayes' theorem.
- For arbitrary events $A$ and $B$, $P(A \cup B) = P(A \setminus B) + P(B \setminus A) + P(A \cap B)$.
- For statistically independent RVs $X$ and $Z$, the following hold:
  1. $\mathbb{E}[X + Z] = \mathbb{E}[X] + \mathbb{E}[Z]$
  2. $\text{var}[X + Z] = \text{var}[X] + \text{var}[Z]$
• (*) The Bienaymé formula: If the RVs $X_i$, $i = 1, \ldots, n$ are uncorrelated, i.e. statistically independent, the variance of their sum is the sum of their variances:
$$\text{var} \left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} \text{var} [X_i].$$

• (*) That for uncorrelated RVs $X_i$, all with variance $\sigma^2$, the variance of the mean of their sum obeys
$$\text{var} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var} [X_i] = \frac{\sigma^2}{n}.$$

• (*) For correlated RVs $X_i$, the variance of their sum obeys
$$\text{var} \left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov} [X_i, X_j].$$

• (*) That for correlated RVs $X_i$ with an average correlation of $\rho$, all with variance $\sigma^2$, the variance of the mean of their sum obeys
$$\text{var} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{\sigma^2}{n} + \frac{2n-1}{n} \rho \sigma^2.$$

• Suppose that we have three coloured boxes $r$ (red), $b$ (blue), and $g$ (green). Box $r$ contains 10 apples, 4 oranges, and 3 limes, box $b$ contains 6 apples, 1 orange, and 2 limes, and box $g$ contains 1 apples, 1 orange, and 8 limes. If a box is chosen at random with probabilities $p(r) = 0.2$, $p(b) = 0.1$, $p(g) = 0.7$, and a piece of fruit is removed from the box (with equal probability of selecting any of the items in the box), then what is the probability of selecting an apple? If we observe that the selected fruit is in fact an orange, what is the probability that it came from the green box?

• – Scenario 1: We roll two fair, six-sided dice. You know that at least one roll resulted in a 6. What is the probability that both rolls were sixes?
• – Scenario 2: Let’s assume we have black dice and white dice. We roll two fair, six-sided dice. One of them results in a 6 and it was a black dice. For the first part, assume that the chance of selecting a black dice is $\frac{1}{2}$. What is the probability that the second dice also rolled a 6? What is the probability if the chance of selecting black dice is $p_b$?
• – (*) What is the formula for three dice rolling sixes if we know that at least one was black and rolled a 6? What about $n$ dice?