Exercise 2

1. In the two category case, under Bayes decision rule the conditional error is given by

\[ P(error|x) = \min[P(\omega_1|x), P(\omega_2|x)] \]  \hspace{1cm} (1)

This form of the conditional error will almost always lead to a discontinuous integrand when calculating the full error by:

\[ P(error) = \int_{-\infty}^{\infty} P(error, x) dx = \int_{-\infty}^{\infty} P(error|x)p(x) dx \]  \hspace{1cm} (2)

Show that for arbitrary densities, we can replace Eq. 1 by

\[ P(error|x) = 2P(\omega_1|x)P(\omega_2|x) \]  \hspace{1cm} (3)

in the integral and get an upper bound on the full error. Show that if we use \( P(error|x) = \alpha P(\omega_1|x)P(\omega_2|x) \) for \( \alpha < 2 \) then we are not guaranteed that the integral gives an upper bound on the error.

2. Suppose that we replace the deterministic function \( \alpha(x) \) with a randomized rule, namely one giving the probability \( P(\alpha_i|x) \) of taking action \( \alpha_i \) on observing \( x \).

Show that the resulting risk is given by

\[ R = \int \left[ \sum_{i=1}^{\alpha} R(\alpha_i|x)P(\alpha_i|x) \right] p(x) dx. \]  \hspace{1cm} (4)

In addition show that \( R \) is minimized by choosing \( P(\alpha_i|x) = 1 \) for the action \( \alpha_i \) associated with the minimum conditional risk \( R(\alpha_i|x) \), and thus no benefit can be obtained from randomizing the best decision rule.

Please send your solutions to gchrupala@lsv.uni-saarland.de by Nov 27.