4. Filtering and Smoothing
Goals of this lecture

- Introduce the basic concept of filters for
  - Time series
  - Images
- Show smoothing of images
General Linear Filter

Filter in the time domain

\[ y_n = \sum_{k=0}^{M} c_k x_{n-k} + \sum_{j=0}^{N-1} d_j y_{n-j-1} \]

- \( x_n \): original signal in the time domain
- \( y_n \): filtered signal
- \( c_k \) and \( d_j \): filter coefficients
Example: Exponential Smoothing

\[ y_n = \alpha x_n + (1 - \alpha) y_{n-1} \]

\( \alpha \): parameter of filter

In principle all samples before the present one influence the output.

What happens to a rect-signal?
Special Cases:

- **FIR-Filter** (finite impulse response)
  \[ y_n = \sum_{k=0}^{M} c_k x_{n-k} \]

- **IIR-Filter** (infinite impulse response)
  \[ y_n = \sum_{k=0}^{M} c_k x_{n-k} + \sum_{j=0}^{N-1} d_j y_{n-j-1} \quad \text{with } N > 0 \]
  - Can have infinite impulse response
  - Also called recursive filters
General *Linear* Filtering of Images

- Convolve image $I_1$ with kernel $K$

\[ I_2 = K \ast I_1 \]

\[ I_2(i, j) = \sum_h \sum_k K(h, k)I_1(i - h, j - k) \]

- $K$ is a finite matrix
- Usually indices of matrix can be negative
Noise Filtering: Mean Smoothing

- Takes a weighted average of the neighborhood.
- For a 3x3 we have

\[
K = \frac{1}{9} \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

- Also referred to as the “box filter”
Noise Filtering: Gaussian Smoothing

- Linear filter, low-pass filter
- Based upon Gaussian distributions

Typically use a 5x5 (min and max)
  - covers 98.8% of the area of the Gaussian when $\sigma=1$ pixel
Separability of the 2-D Gaussian

Separability: kernel can be written as product of one dimensional kernels

For Gaussian kernel:

$$I_2(i, j) = \sum_h \sum_k G(h, k) I_1(i - h, j - k)$$

$$= \sum_h \sum_k e^{-\frac{h^2 + k^2}{2\sigma^2}} I_1(i - h, j - k)$$

**Advantage?**

**Disadvantage?**

$$= \sum_h e^{-\frac{h^2}{2\sigma^2}} \sum_k e^{-\frac{k^2}{2\sigma^2}} I_1(i - h, j - k)$$
Synthetic Example: Gaussian Smoothing

Original image

After filtering with Gaussian kernel

With noise
Realistic Examples

With “salt-and-pepper” noise

⇒ blurring of the image
⇒ use other filters (e.g. median filter)

Gaussian 3x3-kernel
• See maple example
Summary

• Basic definitions of linear filters
  • Time domain
    • FIR filter
    • IIR filter

• Images:
  • Convolution with kernels

• Not covered:
  • How to design a filter